# Linear Algebra I

## 22/01/2019, Tuesday, 14:00 – 17:00

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You are **NOT** allowed to use any type of calculators.

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## 1 Systems of linear equations $(1+4+3+(1+3+3) = 15$ pts)

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Consider the following system of linear equations in the unknowns $v$, $w$, $x$, $y$, and $z$ where $\alpha$ is a real number:

$$\begin{array}{rcrcrcrcrcl} 2v & + & 2w & + & 2x & + & 4y & & & = & 2 \\ & & 2w & + & 4x & + & 2y & + & 2z & = & 0 \\ 4v & + & w & + & & & 5y & + & z & = & 4 \\ 6v & + & 3w & + & 2x & + & 9y & + & z & = & \alpha \end{array}$$

(a) Write down the augmented matrix.

(b) By performing elementary row operations, put the augmented matrix into row echelon form.

(c) Determine all values of $\alpha$ so that the system is consistent.

(d) For the values of $\alpha$ found above,

(i) determine the *lead* and *free* variables.

(ii) put the augmented matrix into *reduced* row echelon form by performing elementary row operations.

(iii) find the solution set.

## 2 Determinants $(9+6 = 15$ pts)

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Consider the matrix

$$M = \begin{bmatrix} \alpha & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

where $\alpha$ is real number.

(a) By using *only* row/column operations, find the determinant of $M$.

(b) Determine all values of $\alpha$ such that $M$ is nonsingular.

## 3 Least squares problem (15 pts)

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Find the line of the form $y = a + bx$ that gives the best least squares approximation to the points:

| $x$ | 1 | 1 | 0 |
|---|---|---|---|
| $y$ | 0 | 1 | 1 |

## 4 Vectorspaces

((3 + 4) + (4 + 4) = 15 pts)

(a) Let $J$ be the $3 \times 3$ matrix given by

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(i) Is the set $S = \{A \in \mathbb{R}^{3\times 3} \mid AJ = JA\}$ a subspace?

(ii) If it is so, find a basis for $S$ and determine its dimension.

(b) Let $M$ be a $2 \times 2$ matrix.

(i) Let $L_M : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ given by $L_M(X) = MX + XM$. Show that $L_M$ is a linear transformation.

(ii) Take $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Write down the matrix representation of $L_M$ using the following basis for $\mathbb{R}^{2\times 2}$:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

## 5 Characteristic polynomial, determinant, and trace

(3 + 4 + 4 + 4 = 15 pts)

Let $M$ be the $3 \times 3$ matrix given by

$$M = \begin{bmatrix} a & b & c \\ a & b & c \\ a & b & c \end{bmatrix}$$

where $a$, $b$, and $c$ are real numbers.

(a) By using the relationship between the determinant and eigenvalues of a matrix, show that 0 is an eigenvalue of $M$.

(b) By using the definition of eigenvalue, show that $a + b + c$ is an eigenvalue of $M$.

(c) By using the relationship between the trace and eigenvalues of a matrix, show that the characteristic polynomial of $M$ is given by $p_M(\lambda) = \lambda^2(\lambda - a - b - c)$.

(d) Suppose that $a + b + c \neq 0$. Show that $M$ is diagonalizable.

## 6 Eigenvalues/vectors

(4 + 5 + 2 + 4 = 15 pts)

Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of $M$.

(b) Show that $M$ is diagonalizable.

(c) Find a matrix $X$ that diagonalizes $M$.

(d) Find $e^M$ by using the matrix $X$.

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10 pts free